

On Some Exact Values of Three-Color Ramsey Numbers for Paths

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Abstract

For graphs G_1, G_2, G_3 , the three-color Ramsey number $R(G_1, G_2, G_3)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with 3 colors, then it contains a monochromatic copy of G_i in color i , for some $1 \leq i \leq 3$.

First, we prove that the conjectured equality $R_3(C_{2n}, C_{2n}, C_{2n}) = 4n$, if true, implies that $R_3(P_{2n+1}, P_{2n+1}, P_{2n+1}) = 4n + 1$ for all $n \geq 3$. We also obtain two new exact values $R(P_8, P_8, P_8) = 14$ and $R(P_9, P_9, P_9) = 17$, furthermore we do so without help of computer algorithms. Our results agree with a formula $R(P_n, P_n, P_n) = 2n - 2 + (n \bmod 2)$ which was proved for sufficiently large n by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007. This provides more evidence for the conjecture that the latter holds for all $n \geq 1$.

1 Definitions

In this paper all graphs are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. For any edge coloring F of a complete graph, F^i will denote the graph induced by the edges of color i in F . Let P_k (resp. C_k) be the path (resp. cycle) on k vertices. The *circumference* $c(G)$ of a graph G is the length of its longest cycle.

The *Turán number* $T(n, G)$ is the maximum number of edges in any n -vertex graph that does not contain any subgraph isomorphic to G . A graph on n vertices is said to be *extremal with respect to G* if it does not contain a subgraph isomorphic to G and has exactly $T(n, G)$ edges.

For given graphs $G_1, G_2, \dots, G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, \dots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n , K_n , with k colors, then it contains a monochromatic copy of G_i in color i , for some $1 \leq i \leq k$. A coloring of the edges of the K_n with k colors is called a $(G_1, G_2, \dots, G_k; n)$ -coloring, if it does not contain a subgraph isomorphic to G_i in color i for any $1 \leq i \leq k$. In the diagonal cases of $G = G_i$ we will write $R_k(G) = R(G_1, G_2, \dots, G_k)$. Finally, we will refer to the first three colors of such Ramsey colorings as red, blue and green, respectively.

2 Overview

In this article we study the values of three-color diagonal Ramsey numbers for paths. In the case of two color Ramsey numbers, a well known theorem of Gerencsér and Gyárfás [5] states that $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ for $n \geq m \geq 2$.

Clearly, we trivially have $R_3(P_1) = 1$ and $R_3(P_2) = 2$. The cases $R_3(P_3) = 5$ and $R_3(P_4) = 6$ are easy but need some thought, while the results $R_3(P_5) = 9$, $R_3(P_6) = 10$ and $R_3(P_7) = 13$ already required help of computer algorithms (see section 6.4.1 of [9] for details and references to these and other related cases). The first open cases are those of $R_3(P_8)$ and $R_3(P_9)$, which are determined later in this paper. All known values agree with a very remarkable result obtained by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007 [8] formulated as follows.

Theorem 1 ([8]) *For all sufficiently large n , we have*

$$R_3(P_n) = \begin{cases} 2n - 1 & \text{for odd } n, \\ 2n - 2 & \text{for even } n. \end{cases} \quad (1)$$

The proof of Theorem 1 is very long and complicated. Our attempts to extract from it any reasonable bound on how large n should be for (1) to hold, failed. Actually, Faudree and Schelp [4], already in 1975, stated that “they feel” that (1) holds for all n . They did so when considering more general cases of $R(P_m, P_n, P_k)$ for paths of different lengths. We believe that the diagonal case deserves the status of a conjecture.

Conjecture 2 ([4]) $R_3(P_n) = 2n - 2 + (n \bmod 2)$ holds for all $n \geq 1$.

The appropriate critical colorings without monochromatic P_n are known for all $n \geq 1$. For $n \geq 2$, one can obtain them by using a “blow-up” of factorization of K_4 (see [8]).

It is interesting to see (1) in the context of the conjectured values of three-color diagonal Ramsey numbers for cycles.

Conjecture 3 ([2][1])

$$R_3(C_n) = \begin{cases} 4n - 3 & \text{for odd } n \geq 5, \\ 2n & \text{for even } n \geq 6. \end{cases} \quad (2)$$

The odd case was conjectured by Bondy and Erdős in 1981 [2], while the even case by the second author in 2005 [1]. Like with (1) for paths, (2) is known to hold for all sufficiently large n , and the first open cases are those of $R_3(C_9)$ and $R_3(C_{10})$ (see section 6.3.1 of [9] for details and references to other related cases and asymptotics).

In section 3 we will prove an interesting implication that the even n case of (2) implies the odd $(n + 1)$ case of (1) for $n \geq 6$. The equalities $R_3(C_6) = 12$ [12] and $R_3(C_8) = 16$ [10] were obtained with the help of computer algorithms. Thus, it will imply that $R_3(P_7) = 13$ and $R_3(P_9) = 17$. We will also provide a computer-free proof of the latter. Finally, we prove that $R_3(P_8) = 14$, which leaves $R_3(P_{10})$ as the first open case of (1).

3 Related Background Results

Gyárfás, Rousseau and Schelp [7] completely solved the question of what is the maximum number of edges $f(m, n, k)$ in any P_k -free subgraph of the complete bipartite graph $K_{m,n}$. They also characterized all the corresponding extremal graphs. Tables III and IV in [7] present formulas for $f(m, n, k)$ for even and odd k , respectively, and Tables I and II therein describe the constructions of all the extremal graphs achieving $f(m, n, k)$. In the proofs of sections 4 and 5 we will refer to these tables several times.

Also in the proofs we will need some values of Turán numbers for paths. In order to determine the required $T(n, P_k)$, the following theorem by Faudree and Schelp will be used.

Theorem 4 ([4]) *If G is a graph with $|V(G)| = kt + r$, $r < k$, $0 \leq t, r$, containing no P_{k+1} , then $|E(G)| \leq t \binom{k}{2} + \binom{r}{2}$ with equality if and only if G is either $(tK_k) \cup K_r$ or $((t-l-1)K_k) \cup (K_{(k-1)/2} + \overline{K}_{(k+1)/2+lk+r})$ for some $0 \leq l < t$ when k is odd, $t > 0$, and $r = (k \pm 1)/2$.*

The following notation and terminology comes from [3]. For positive integers a and b , define $r(a, b)$ as

$$r(a, b) = a - b \left\lfloor \frac{a}{b} \right\rfloor = a \bmod b.$$

For integers $n \geq k \geq 3$, define $w(n, k)$ by

$$w(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1), \quad (3)$$

where $r = r(n-1, k-1)$.

Woodall's theorem [11] can then be formulated as follows.

Theorem 5 ([3]) *Let G be a graph on n vertices and m edges with $m \geq n$ and $c(G) = k$. Then*

$$m \leq w(n, k),$$

and this result is best possible.

In [3], one can find the description of all extremal graphs achieving $w(n, k)$.

4 Progress on $R_3(P_{2n+1})$

First we prove the following general implication.

Theorem 6 *For all $n \geq 3$, if $R_3(C_{2n}) = 4n$, then $R_3(P_{2n+1}) = 4n + 1$.*

Proof. The lower bound follows from the “blow-up” construction (see [8]).

For the upper bound, suppose that there exists a 3-edge coloring of K_{4n+1} without monochromatic P_{2n+1} . From the assumption that $R_3(C_{2n}) = 4n$, we know that this coloring contains a monochromatic C_{2n} . Without loss of generality, we assume that it is red. Now, in order to avoid red P_{2n+1} no vertex on this cycle can be connected by a red edge to any vertex outside of the cycle. Hence, we have a complete bipartite graph $K_{2n,2n+1}$ with only blue and green edges. Let the parts of this bipartite graph be called X (vertices on the cycle) and Y (vertices outside of the cycle). Using the notation of [7], we have

$$a = 2n = |X|, \quad b = 2n + 1 = |Y|, \quad c = n - 1, \quad a = 2(c + 1),$$

hence, if we apply the last row of Table IV in [7], then we obtain

$$f_1(a, b, c) = (a + b - 2c)c = 2n^2 + n - 3.$$

This implies that

$$2f_1(a, b, c) = 4n^2 + 2n - 6 < 2n(2n + 1) = |X||Y|,$$

and therefore blue and green edges cannot account for all the edges of $K_{X,Y}$ without creating a monochromatic P_{2n+1} . This completes the proof of the upper bound $R_3(P_{2n+1}) \leq 4n + 1$. \square

Corollary 7 $R_3(P_7) = 13$ and $R_3(P_9) = 17$.

Proof. It is known that $R_3(C_6) = 12$ [12] and $R_3(C_8) = 16$ [10]. By Theorem 6, these imply that $R_3(P_7) = 13$ and $R_3(P_9) = 17$. \square

The upper bounds in $R_3(C_6) = 12$ and $R_3(C_8) = 16$ were obtained with the help of computer algorithms. In the proof of the next theorem we provide a computer-free proof of the upper bound $R_3(P_9) \leq 17$. The proof of $R_3(P_7) \leq 13$ can be obtained by a similar reasoning.

Theorem 8 (*computer-free*)

$$R_3(P_9) \leq 17.$$

Proof. We need to show that each 3-coloring of the edges of K_{17} contains a monochromatic P_9 . Let us suppose that there is a $(P_9, P_9, P_9; 17)$ -coloring G with colors red, blue and green, forming graphs G^1 , G^2 and G^3 , respectively. Since K_{17} has 136 edges, we may assume without loss of generality that there are at least 46 red edges, i.e. $e(G^1) \geq 46$.

Since by (3) we have $w(17, 6) = 46$, it follows by Theorem 5 that G^1 contains a cycle C_k for some $k \geq 6$. One can easily verify that the critical graphs in this case [3] have P_9 , and thus $k \geq 7$. If $k \geq 9$, we immediately obtain a red P_9 , a contradiction. If $k = 8$, then to avoid a P_9 in G^1 we have a bipartite graph G' with partite sets of order 8 and 9, respectively. In order to avoid monochromatic P_9 in G^2 and G^3 , G' contains at most 66 blue and green edges (use the last row of Table IV in [7]). Each of at least 6 other edges of G' are red and together with the C_8 they contain a red P_9 , again a contradiction. Hence, in the rest of the proof we will assume that G has a red C_7 with vertices $C = \{c_1, c_2, \dots, c_7\}$, and the remaining vertices are $P = \{p_1, p_2, \dots, p_{10}\}$.

Claim. *Let H be a 3-coloring of the edges of K_{17} , and suppose that $P_9 \not\subset H^i$ for $1 \leq i \leq 3$, $|H^1| \geq 46$, $c(H^1) = 7$, and H^1 contains a C_7 . Then there are at least 4 vertices in $V(H) \setminus V(C_7)$ joined by at least one red edge to the cycle C_7 .*

Proof of the Claim. Consider the coloring H as stated above. Let the vertices of C_7 in H^1 be $C = \{c_1, c_2, \dots, c_7\}$, and the remaining vertices of H are $P = \{p_1, p_2, \dots, p_{10}\}$. We will use the tables in [7] several times when considering bipartite subgraphs of $K_{|C|,k} = K_{7,k}$ for $k = 10, 9, 8$ and 7 . We prove that there are red edges in these bipartite subgraphs, and so for each k we obtain one more vertex in $V(H) \setminus C$ joined to the cycle C_7 by at least one red edge.

The maximum possible number of edges in a bipartite graph with partite sets 7 and $k = 10$ or 9 without P_9 is $7 + 3(k - 1)$, which follows from Table IV in [7] with $a = 7$, $b = k$, $c = 3$ and $f_1(a, b, c) = a + (b - 1)c$. Since $2(7 + 3(k - 1)) < 7k = e(K_{7,k})$ for $k = 10$ and 9 , we obtain the first 2 vertices, say p_1 and p_2 , connected to C_7 by at least one red edge.

Now, we consider the bipartite graph $K_{|C|, |P \setminus \{p_1, p_2\}|} = K_{7,8}$. Similarly to the previous case, the maximum number of edges in this bipartite graph without P_9 is $7 + 3(8 - 1) = 28$. This time, however, this is exactly half of the edges of $K_{7,8}$, so we need to consider the possible extremal graphs. By Table II in [7] these extremal graphs are G_{14} and G_{15} with $a = 7$, $b = 8$ and $c = 3$, and they can be eliminated as follows:

- $G_{14} = K_{7,8} - K_{4,7}$. Clearly, $K_{4,7}$ contains a P_9 , so it cannot consist of the edges of single color, a contradiction.
- $G_{15} = K_{4,4} \cup K_{3,4}$, which under bipartite complement is isomorphic to itself. Let us consider vertex p_1 . To avoid red C_8 the vertex p_1 is joined by at most 3 red edges to the cycle C_7 . By considering the remaining edges from the vertex p_1 we see that at least one, say blue, is connected to the $K_{4,4}$ part of G_{15} in blue. This easily gives a monochromatic P_9 , a contradiction.

Thus, we have the third vertex, say p_3 , connected to C by a red edge.

Finally, let us consider the bipartite graph $K_{C, P \setminus \{p_1, p_2, p_3\}}$. By the third case in Table IV of [7], the maximum possible number of edges in this bipartite graph without P_9 is $7 + 3(7 - 1) = 25$. By Table II, there are two possible extremal graphs: G_{14} and G_{15} which now are $K_{7,7} - K_{4,6}$ and $K_{4,4} \cup K_{3,3}$, respectively. By considering possible edges from p_1, p_2 and p_3 to the cycle C_7 , similarly as for $k = 8$, we have a monochromatic P_9 , so we obtain the fourth required vertex p_4 . This completes the proof of the Claim.

We have $m \geq 4$ vertices $M = \{p_1, \dots, p_m\} \subset P$ not on the red C_7 joined to it by some red edges. Note that any red edge $\{p_i, p_j\}$ prevents p_i and p_j to be connected by any red edge to C , hence P induces at most $\binom{10-m}{2}$ red edges. To avoid red C_8 , the vertices in M can be joined by at most 3 red edges each to C (to vertices nonadjacent on the cycle C_7).

First, consider the case when a vertex in M , say p_1 , has 3 red edges to C , wlog $\{p_1, c_1\}$, $\{p_1, c_3\}$ and $\{p_1, c_5\}$. Note that no vertex $p \in M$, $p \neq p_1$, can be joined by any red edge to the vertices in the set $\{c_2, c_4, c_6, c_7\}$, since otherwise a red P_9 from p to p_1 can be easily constructed. In addition, if there is the red edge $\{p_2, c_5\}$, then the edges $\{c_4, c_6\}$, $\{c_4, c_7\}$, $\{c_2, c_4\}$ are blue or green. For example, if $\{c_4, c_6\}$ is red, then $p_2 c_5 c_4 c_6 c_7 c_1 c_2 c_3 p_1$ is a red P_9 . Similarly, if $\{p_2, c_1\}$ or $\{p_2, c_3\}$ is red, then at least three edges induced

in C must be blue or green. In all cases, C induces at most 18 red edges. Thus, counting red edges in C , between C and P , and in P , we have

$$e(G^1) \leq 18 + 3m + \binom{10-m}{2}. \quad (4)$$

Observe that the set $M \cup \{c_2, c_4, c_6, c_7\}$ induces only blue and green edges, hence $R(P_9, P_9) = 12$ [5] implies that $m + 4 \leq 11$. By the Claim we have $m \geq 4$, so $4 \leq m \leq 7$, and we find that $e(G^1) < 46$ for all possible m . This is a contradiction.

Finally we consider the case when all vertices in M are connected to C by at most 2 red edges. Counting again red edges, for all possible $4 \leq m \leq 10$, we obtain

$$e(G^1) \leq \binom{7}{2} + 2m + \binom{10-m}{2} < 46, \quad (5)$$

which is a contradiction. \square

5 $R_3(P_8) = 14$

We begin with a lemma which is technically very similar to the claim within the proof of theorem 8.

Lemma 9 *Let H be a 3-coloring of the edges of K_{14} , and suppose that $P_8 \not\subset H^i$ for $1 \leq i \leq 3$, $|H^1| \geq 31$, $c(H^1) = 6$, and H^1 contains a C_6 . Then there are at least 3 vertices in $V(H) \setminus V(C_6)$ joined by at least one red edge to the cycle C_6 .*

Proof. We give only the sketch of proof because the details are very similar to those in the proof of Claim in Theorem 8. By using three times Tables I and III in [7] and considering bipartite subgraphs $K_{6,k}$ for $k = 8, 7, 6$, we obtain that the maximum number of edges in these bipartite subgraphs without P_8 is $3k$. From Table I, the extremal graphs are $K_{3,l} \cup K_{3,k-l}$, where $0 \leq l \leq k$. By considering the remaining edges of H , one can easily obtain a monochromatic P_8 in all cases, a contradiction. \square

Theorem 10

$$R_3(P_8) = 14.$$

Proof. We need to show that every 3-edge coloring of K_{14} contains a monochromatic P_8 . Let us suppose that there is a $(P_8, P_8, P_8; 14)$ -coloring G with colors red, blue and green, forming graphs G^1 , G^2 and G^3 , respectively. Since K_{14} has 91 edges, we may assume without loss of generality that there are at least 31 red edges, i.e. $e(G^1) \geq 31$.

Since by (3) we have $w(14, 5) = 31$, it follows by Theorem 5 that G^1 contains a cycle C_k for some $k \geq 5$. One can easily verify that the critical graphs in this case (see [3]) have P_8 , and thus $k \geq 6$. If $k \geq 8$, then we immediately obtain a P_8 , a contradiction. If $k = 7$, then to avoid a P_8 in G^1 we have a bipartite graph G' with two partite sets of order 7. In order to avoid monochromatic P_8 in G^2 and G^3 , the graph G' contains at most 48 blue and green edges (use row 3 in Table III in [7]). At least one remaining edge of G' is red and together with the C_7 we have a red P_8 , a contradiction. Hence, in the rest of the proof we will assume that G has a red C_6 with vertices $C = \{c_1, c_2, \dots, c_6\}$, and the remaining vertices are $P = \{p_1, p_2, \dots, p_8\}$.

By Lemma 9 we have $m \geq 3$ vertices $M = \{p_1, \dots, p_m\} \subset P$ not on the red C_6 joined to it by some red edges. Note that any red edge $\{p_i, p_j\}$ prevents p_i and p_j to be connected by any red edge to C , hence P induces at most $\binom{8-m}{2}$ red edges. To avoid red C_7 , the vertices in M can be joined by at most 3 red edges each to C (to vertices nonadjacent on the cycle C_6). We will be counting red edges in C , between C and P , and in P , similarly as in (4) and (5).

First, consider the case when all the vertices in M are connected to C by at most 2 red edges each. If at least one them is connected to 2 vertices in C , then at least one of the edges induced by C is not red, or there are less than $2m$ edges between C and P . Hence, for all possible $3 \leq m \leq 8$, we have

$$e(G^1) \leq \binom{6}{2} + 2m - 1 + \binom{8-m}{2} < 31,$$

which gives a contradiction.

The remaining case is when some vertex in M is connected to C by exactly 3 red edges, say p_1 , and the red edges from p_1 to C are $\{p_1, c_1\}$, $\{p_1, c_3\}$, $\{p_1, c_5\}$. Then no vertex $p_i \in P$, $2 \leq i \leq 8$, can be joined by a red edge to any of the vertices in the set $\{c_2, c_4, c_6\}$. In addition, if there is the

red edge $\{p_2, c_1\}$, then the edges $\{c_2, c_4\}$, $\{c_2, c_6\}$, $\{c_4, c_6\}$ are blue or green. For example, if $\{c_2, c_4\}$ is red, then $p_2c_1c_2c_4c_3p_1c_5c_6$ is a red P_8 . Similarly, if $\{p_2, c_3\}$ or $\{p_2, c_5\}$ is red, then at least the same three edges induced in C must be blue or green. In all cases, C induces at most 12 red edges.

Observe that the set $M \cup \{c_2, c_4, c_6\}$ has only blue and green edges, hence $R(P_8, P_8) = 11$ [5] implies that $m + 3 \leq 10$. Note that if $m = 7$, then the sole vertex in $P \setminus M$ is not in any red edge, so we can decrease the range of m further to $3 \leq m \leq 6$. This time we obtain

$$e(G^1) \leq 12 + 3m + \binom{8-m}{2}. \quad (6)$$

$e(G^1)$ can achieve 31 in (6) for $m = 3$ and $m = 6$, furthermore only in cases when all (3 or 6) vertices in M are connected by exactly 3 red edges to C . We will show that in both cases G has a blue or green P_8 .

If $m = 6$, then the equality in (6) implies that $P \cup \{c_2, c_4, c_6\}$ contains exactly one red edge between 2 vertices in $P \setminus M$, or equivalently, the $K_{11} - e$ with vertices $P \cup \{c_2, c_4, c_6\}$ has all its 54 edges blue or green. By Theorem 4 with $k = 7$, $t = 1$ and $r = 4$ we obtain $T(11, P_8) = 27$. One can easily check that it is not possible for two copies of the corresponding extremal graphs to cover $K_{11} - e$.

The last situation to consider is that of $m = 3$, where G^1 has two components: one spanned by 9 vertices of $C \cup M$ with 21 red edges and a red K_5 on vertices $Q = P \setminus M = \{p_4, \dots, p_8\}$. The set $H = M \cup \{c_2, c_4, c_6\}$ has no red edges. Denote by R the set $\{c_1, c_3, c_5\}$. The 60 edges of $G^2 \cup G^3$ form a complete K_6 on H and two complete bipartite graphs $K_{H,Q}$ and $K_{Q,R}$. Let P_l be the longest monochromatic, say blue, path in H , and denote by a and b its endpoints. By Theorem 4 we have $T(6, P_4) = 6$, which implies that $l = 6$ or $l = 5$. We have the following possibilities:

Case 1. There are no blue edges joining a or b to Q (for $l = 5$ or $l = 6$).

We have $H \cup R = C \cup M$, and let $S = C \cup M \setminus \{a, b\}$. We consider the complete bipartite graph $K_{5,7}$ with partite sets Q and S . Because all the edges from a and b to Q are green, this $K_{Q,S}$ cannot have green P_4 . The third row of Table III in [7], with $a = 5$, $b = 7$ and $c = 1$, implies that there are at most 10 green edges between Q and S . Clearly, $K_{Q,S}$ cannot have blue P_8 . We now use the second row of the same Table III with $c = 3$, and see that there are at most 21 blue edges between Q and S . There are not enough green and blue edges to cover all 35 edges of $K_{Q,S}$, which is a contradiction.

Case 2. There is a blue edge from a to Q , say $\{a, p_4\}$, and $l = 6$.

Let the blue P_l in H be $as_1s_2s_3s_4b$. If there is no blue P_8 , then all the edges joining b to p_i , $5 \leq i \leq 8$, and joining p_4 to R are green. We consider the colors of the edges from s_4 to the set $Q \setminus \{p_4\} = \{p_5, p_6, p_7, p_8\}$. This case is now broken into three subcases, as follows:

1. There are at least two blue edges from s_4 to $Q \setminus \{p_4\}$, say $\{s_4, p_5\}$ and $\{s_4, p_6\}$. To avoid blue P_8 all the edges between $\{p_5, p_6\}$ and R must be green, but in this case we have a green $P_8 = p_8bp_5c_1p_6c_3p_4c_5$.
2. There is exactly one such blue edge, say $\{s_4, p_5\}$. To avoid blue P_8 all the edges between p_5 and R must be green, but then we have a green $P_8 = c_1p_4c_3p_5bp_6s_4p_7$.
3. All edges from s_4 to $\{p_5, p_6, p_7, p_8\}$ are green. If there is a green edge between R and $\{p_5, p_6, p_7, p_8\}$, say $\{c_1, p_5\}$, then we have a green $p_7s_4p_6bp_5c_1p_4c_3$. So, assume that all edges from R to $\{p_5, p_6, p_7, p_8\}$ are blue. If there is at least one blue edge from $\{p_5, p_6, p_7, p_8\}$ to $\{s_2, s_3\}$, say $\{p_5, s_2\}$, then we have a blue $p_4as_1s_2p_5c_1p_6c_3$. In the opposite case we obtain a green $p_4s_4p_5s_3p_6s_2p_7b$.

Case 3. There is a blue edge from a to Q , say $\{a, p_4\}$, all the edges from b to $Q \setminus \{p_4\}$ are green, and $l = 5$ (the edge $\{b, p_4\}$ can be blue or green).

Let the blue P_l in H be $as_1s_2s_3b$. There is a vertex $c \in H$, such that the edges $\{a, c\}$ and $\{b, c\}$ are green, since otherwise $l = 6$. This case is broken into three subcases, as follows:

1. There are at least two blue edges from p_4 to R , say $\{p_4, c_1\}$ and $\{p_4, c_3\}$. When avoiding blue P_8 , we obtain a green $P_8 = acbp_8c_3p_7c_1p_6$.
2. There is exactly one blue edge from p_4 to R , say $\{p_4, c_1\}$. Then, if there is at least one green edge from $\{c_3, c_5\}$ to $Q \setminus \{p_4\}$, say $\{c_3, p_5\}$, then we have green $P_8 = acbp_8c_1p_5c_3p_4$. In the opposite case we must have a blue complete bipartite subgraph $K_{\{c_3, c_5\}, \{p_5, p_6, p_7, p_8\}}$. If there is at least one blue edge from $\{p_5, p_6, p_7, p_8\}$ to $\{s_1, s_2, s_3\}$, then we have a blue P_8 , otherwise we easily find a green P_8 .

3. All the edges from p_4 to R are green. Then, if there is at least one green edge from R to $Q \setminus \{p_4\}$, say $\{c_1, p_5\}$, then in order to avoid a green P_8 , we must have a blue complete bipartite $K_{\{c_3, c_5\}, \{p_6, p_7, p_8\}}$. In the opposite case, we have a complete blue bipartite subgraph $K_{\{c_1, c_3, c_5\}, \{p_6, p_7, p_8\}}$. If there is at least one blue edge from $\{p_6, p_7, p_8\}$ to $\{s_1, s_2, s_3\}$, then we have a blue P_8 , otherwise we have a green P_8 .

Case 4. There is a blue edge from a to Q , say $\{a, p_4\}$, there is a blue edge from b to a different vertex in Q , say $\{b, p_8\}$, and $l = 5$.

Let the blue P_l in H be $as_1s_2s_3b$. There is a vertex $c \in H$, such that the edges $\{a, c\}$ and $\{b, c\}$ are green, since otherwise $l = 6$. All the edges from $\{p_4, p_8\}$ to $R \cup \{c\}$ are green. If there are at least two green edges from a vertex in $\{p_5, p_6, p_7\}$ to R , say $\{p_5, c_1\}$ and $\{p_5, c_3\}$, then we have a green P_8 , namely $c_5p_4c_1p_5c_3p_8cb$. In the opposite case, we have a blue P_4 , without loss of generality, say $c_1p_5c_3p_6$. To avoid a blue P_8 , the edges $\{a, p_6\}$ and $\{s_1, p_6\}$ are green, but then we have a green $P_8 = s_1p_6acp_4c_1p_8c_5$. \square

It is interesting to observe that the smaller case of $R_3(P_8)$ required significantly more complex reasoning than that of $R_3(P_9)$. In general, we expect that even paths cases are harder than those for odd paths. Consequently, between the first two open cases of Conjecture 2, namely the questions whether it is true that $R_3(P_{10}) = 18$ and $R_3(P_{11}) = 21$, we expect the latter to be simpler to prove.

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